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# On a $p$-Kirchhoff equation via Krasnoselskii's genus

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ABSTRACT

In this work will use the genus theory, introduced by Krasnoselskii, to show a result of existence and multiplicity of solutions of the $p$-Kirchhoff equation

$$-\left[M\left(\int_\Omega |\nabla u|^p \mathrm{d}x\right)\right]^{p-1} \Delta_p u = f(x,u) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega$$

where $\Omega$ is a bounded smooth domain of $\mathbb{R}^N$, $1 < p < N$, and $M$ and $f$ are continuous functions.



## 1. Introduction

In this work we deal with questions of existence and multiplicity of solutions of the $p$-Kirchhoff equation

$$\begin{cases} -\left[M(\|u\|^p)\right]^{p-1} \Delta_p u = f(x,u) & \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega \end{cases} \tag{P}$$

where, throughout this work, $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ and $M : \mathbb{R}^+ \to \mathbb{R}^+$ are continuous functions that satisfy conditions which will be stated later, $\Delta_p u$ is the $p$-Laplacian operator, that is,

$$\Delta_p u = \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(|\nabla u|^{p-2} \frac{\partial u}{\partial x_i}\right), \qquad 1 < p < N,$$

and $\|.\|$ is the usual norm in $W_0^{1,p}(\Omega)$ given by

$$\|u\|^p = \int_\Omega |\nabla u|^p.$$

We assume the following hypotheses for $M$ and $f$:
There are positive constants $A$, $B$ and $\alpha$ such that

$$At^\alpha \le [M(t)]^{p-1} \le Bt^\alpha, \tag{M}$$

and there are positive constants $Q_1$, $Q_2$ and $q$ such that

$$Q_1 t^{q-1} \le f(x,t) \le Q_2 t^{q-1}, \tag{f$_1$}$$

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for all $t \geq 0$ and for all $x \in \overline{\Omega}$, where $q \in (p, p^* = \frac{Np}{N-p})$ and $\alpha > q/p$,

$$f(x,t) = -f(x,-t) \tag{f$_2$}$$

for all $t \in \mathbb{R}$ and for all $x \in \overline{\Omega}$.

We use the genus theory, introduced by Krasnoselskii [8], to prove our main result, as follows:

**Theorem 1.1.** *Assume $(M)$, $(f_1)$ and $(f_2)$. Then $(P)$ has infinitely many solutions.*

Problem $(P)$ was motivated by an example contained in [3,7].

## 2. Preliminary results

We will start by considering some basic notions on the Krasnoselskii genus that we will use in the proof of our main result.

Let $E$ be a real Banach space. Let us denote by $\mathfrak{A}$ the class of all closed subsets $A \subset E \setminus \{0\}$ that are symmetric with respect to the origin, that is, $u \in A$ implies $-u \in A$.

**Definition 2.1.** *Let $A \in \mathfrak{A}$. The Krasnoselskii genus $\gamma(A)$ of $A$ is defined as being the least positive integer $k$ such that there is an odd mapping $\phi \in C(A, \mathbb{R}^k)$ such that $\phi(x) \neq 0$ for all $x \in A$. If such a $k$ does not exist we set $\gamma(A) = \infty$. Furthermore, by definition, $\gamma(\emptyset) = 0$.*

In the sequel we will establish only the properties of the genus that will be used through this work. More information on this subject may be found in the references [2,3,7,8].

**Theorem 2.2.** *Let $E = \mathbb{R}^N$ and $\partial\Omega$ be the boundary of an open, symmetric and bounded subset $\Omega \subset \mathbb{R}^N$ with $0 \in \Omega$. Then $\gamma(\partial\Omega) = N$.*

**Corollary 2.3.** $\gamma(S^{N-1}) = N$.

As a consequence of this, if $E$ is of infinite dimension and separable and $S$ is the unit sphere in $E$, then $\gamma(S) = \infty$.

We now establish a result due to Clarke [4].

**Theorem 2.4.** *Let $J \in C^1(X, \mathbb{R})$ be a functional satisfying the Palais–Smale condition. Furthermore, let us suppose that:*
(i) *$J$ is bounded from below and even;*
(ii) *there is a compact set $K \in \mathfrak{A}$ such that $\gamma(K) = k$ and $\sup_{x\in K} J(x) < J(0)$.*
*Then $J$ possesses at least $k$ pairs of distinct critical points and their corresponding critical values are less than $J(0)$.*

We point out that this result is a consequence of a basic multiplicity theorem involving an invariant functional under the action of a compact topological group.

Let us go back to the problem $(P)$. For this, we consider the functional $J : H_0^1(\Omega) \to \mathbb{R}$ defined by

$$J(u) = \frac{1}{p}\widehat{M}(\|u\|^p) - \int_\Omega F(x,u),$$

where $\widehat{M}(t) = \int_0^t [M(s)]^{p-1} \mathrm{d}s$ and $F(x,t) = \int_0^t f(x,s)\mathrm{d}s$. Plainly, $J \in C^1(W_0^{1,p}(\Omega), \mathbb{R})$ and

$$J'(u)\phi = [M(\|u\|)^p]^{p-1} \int_\Omega |\nabla u|^{p-2} \nabla u \nabla \phi - \int_\Omega f(x,u)\phi,$$

for all $u, \phi \in W_0^{1,p}(\Omega)$.

In the proof of Theorem 1.1 we shall need the following technical results

**Lemma 2.5.** *$J$ is bounded from below.*

**Proof.** Using $(M)$ and $(f_1)$ we get

$$J(u) \geq \frac{1}{p} \int_0^{\|u\|^p} [As^\alpha] \mathrm{d}s - C \int_\Omega |u|^q$$

and by Sobolev immersions

$$J(u) \geq \frac{A}{p(\alpha+1)} \|u\|^{p(\alpha+1)} - C_1 \|u\|^q.$$

Since $\alpha > q/p$ we have $\alpha > \frac{q-p}{p}$ or $p(\alpha+1) > q$. So $J$ is bounded from below. ■

**Lemma 2.6.** *J satisfies the* (*PS*) *condition.*

**Proof.** Let $(u_n)$ be a sequence in $W^{1,p}(\Omega)$ such that

$$J(u_n) \to C \text{ and } J'(u_n) \to 0.$$

Arguing as in the lemma above, we obtain a positive constant $C_1$ such that

$$C_1 \geq J(u_n) \geq \frac{A}{p}\|u_n\|^{p(\alpha+1)} - Q_2 C\|u_n\|^q.$$

Since $\alpha > p/q$, we conclude that $(\|u_n\|)$ is bounded. Thus, passing to a subsequence, if necessary, we have

$$\|u_n\|^p \to t_0 \geq 0.$$

If $t_0 = 0$, then the proof is finished. If $t_0 > 0$ then, since $M$ is a continuous function, we get

$$M(\|u_n\|^p) \to M(t_0).$$

Thus, for $n$ sufficiently large,

$$M(\|u_n\|^p) \geq \overline{C} > 0,$$

for some constant $\overline{C}$. Now, arguing as in [5], we obtain

$$o_n(1) \geq \overline{C}C_p \int_\Omega |\nabla u_n - \nabla u|^p,$$

where $C_p$ is a constant that appears in the standard inequality in $\mathbb{R}^N$ given by

$$\langle |x|^{p-2}x - |y|^{p-2}y, x-y\rangle C_p|x-y|^p$$

if $p \geq 2$ or

$$\langle |x|^{p-2}x - |y|^{p-2}y, x-y\rangle \frac{C_p|x-y|^2}{(|x|+|y|)^{2-p}},$$

if $1 < p < 2$, where $\langle , \rangle$ is the Euclidean inner product in $\mathbb{R}^N$. Thus, we conclude that $\|u_n - u\| \to 0$ in $W^{1,p}(\Omega)$. ■

## 3. Proof of Theorem 1.1

Let us consider $\{e_1, e_2, \ldots\}$, a Schauder basis of $W_0^{1,p}(\Omega)$, and for each $k \in \mathbb{N}$ consider $X_k = span\{e_1, e_2, \ldots, e_k\}$, the subspace of $W_0^{1,p}(\Omega)$ generated by $k$ vectors $e_1, e_2, \ldots, e_k$. Note that $X_k \hookrightarrow L^q(\Omega)$, $1 \leq q \leq p^*$, with continuous immersions. Thus, the norms $W_0^{1,p}(\Omega)$ and $L^q(\Omega)$ are equivalent on $X_k$ and, hence, there exists a positive constant $C(k)$ which depends on $k$, such that

$$-C(k)\|u\|^q \geq -\int_\Omega |u|^q,$$

for all $u \in X_k$. We now use $(M)$ and $(f_1)$ to conclude that

$$J(u) \leq \frac{B}{p(\alpha+1)}\|u\|^{p(\alpha+1)} - C(k)Q_1\|u\|^q,$$

or also

$$J(u) \leq \|u\|^q \left(\frac{B}{p(\alpha+1)}\|u\|^{p(\alpha+1)-q} - C(k)Q_1\right).$$

Let $R$ be a positive constant such that

$$\frac{B}{p(\alpha+1)}R^{p(\alpha+1)-q} < C(k)Q_1.$$

Thus, for all $0 < r < R$, and considering $K = \{u \in X_k : \|u\| = r\}$, we get

$$J(u) \leq r^q\left(\frac{B}{p(\alpha+1)}r^{p(\alpha+1)-q} - C(k)Q_1\right) < R^q\left(\frac{B}{p(\alpha+1)}R^{p(\alpha+1)-q} - C(k)Q_1\right) < 0 = J(0),$$

which implies

$$\sup_K J(u) < 0 = J(0).$$

Since $X_k$ and $\mathbb{R}^k$ are isomorphic and $K$ and $S^{k-1}$ are homeomorphic, we conclude that $\gamma(k) = k$. Moreover, from $(f_2)$, $J$ is even. By the Clarke theorem, $J$ has at least $k$ pairs of different critical points. Since $k$ is arbitrary, we obtain infinitely many critical points of $J$. ■

## 4. Final remarks

Problem $(P)$ is a generalization of the classical Kirchhoff equation

$$\begin{cases} -M(\|u\|^2)\Delta u = f(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{K}$$

where $\|u\|^2 = \int_\Omega |\nabla u|^2$, which is the stationary counterpart of the well known hyperbolic Kirchhoff equation

$$\frac{\partial^2 u}{\partial t^2} - M(\|u\|^2)\Delta u = f(x,u) \quad \text{in } \Omega. \tag{4.1}$$

As regards problems $(P)$ and $(K)$, they have been attacked from the variational point of view, always making the assumption

$$M(t) \geq m_0 > 0 \quad \text{for all } t \geq 0, \tag{4.2}$$

where $m_0$ is a constant. See [1,5,6] for example.

In our case, the function $M$ does not satisfy assumption (4.2). Furthermore, we obtain infinitely many solutions. To our knowledge, these facts are novel for problem $(P)$. We point out that in a recent work [9] the author considers $M$ decreasing and $M(t)$ may go to zero as $t \to +\infty$, that is, once again, $M$ does not obey hypothesis (4.2). In this latter case, the author uses the Mountain Pass Theorem and obtains only one solution.

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